Computational Implementation of Cosserat Based
Strain Gradient Plasticity Theories

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ABSTRACT
The current trend in microelectronics towards miniaturization has pushed for an interest in
theories intended to explain the behavior of materials at small scales.  In particular, an increase in
yield strength with decreasing size has been experimentally observed in several materials and
under different loading conditions.  A class of non-classical continuum mechanics theories has
been recently employed in order to explain the wide range of observed size dependent
phenomena.  The theories are non-classical in the sense that they bring about additional
kinematic variables.  In the numerical treatment of such theories two issues are clearly identified.
First, in a displacement based finite element approach the need appears for higher orders of
continuity in the interpolation functions or else alternative formulations must be used.  Second, if
nonlinear-inelastic material response is expected the theories should be recast in rate form and
the corresponding integration algorithms should complement the implementation.  In this article
we address both problems for the particular case of a Cosserat couple stress theory.  We describe
alternatives for the numerical treatment and then we extend the framework to the case of a rate
independent inelastic - non-linear material behavior.  The equations are presented in its flow
theory form together with integration algorithms.

Keywords: Strain gradient plasticity, size effect, length scale, constitutive modeling, integration
algorithms.
1.0 Introduction.

The development of microelectronics and other small scale related problems have pushed for a recent interest in continuum mechanics theories incorporating length scales. This class of theories is mainly intended to explain size dependent response observed in a variety of materials and testing conditions. Classical continuum mechanics theories have no internal length scale and they implicitly assume that the wavelength of the imposed deformation field is many times larger than the representative volume element (RVE) of the material. This means that there is no deformation localization or that gradients of strain are rapidly smoothed out. Under plastic conditions and non-uniform deformation fields it appears that localization phenomena occur at specimen sizes which are already many times larger than this RVE. In other words, the response becomes size dependent at volumes where continuum mechanics theories are still applicable. Since classical continuum mechanics theories have no internal length scale, they are unable to predict the wide range of observed size dependent phenomena. In the strain gradient theories available in the literature the length scale appear as an additional material parameter that enhances the resistance with the gradients of strain. Theories incorporating length scale ($\ell$) material parameters can be identified in the works of Bazant(1984), Aifantis(1992), Borst and Muhlhaus(1992), Fleck and Hutchinson(1993), Fleck et al(1993), Fleck and Hutchinson(1997), Begley and Hutchinson (1998), Gao et al(1999a), Gao et al(1999b), Guo et al(2001), Bassani(2001), Bazant(2002), Borst and Abellan (2002), Gutierrez and Borst (2002), Abu Al-Rub and Voyiadjis(2004), Gomez and Basaran (2005) and Han et (2005) Within this class of theories, when some characteristic dimension representative of the plastic deformation field of the particular problem approaches the material length scale, size effects are triggered. The present article is not intended to prove the validity of the so-called size effects problem but it
rather intends to discuss its numerical treatment. The interested reader is referred to the literature in the field like, in Fleck and Hutchinson(1997), Gao et al(1999a, 1999b), and more recently Abu Al-Rub and Voyiadjis(2004). The dependence on size has been mainly substantiated by three experiments; the plastic torsion of thin wires, Fleck and Hutchinson(1993), the nano-indentation experiments on metals and ceramics, Stelemashenko et al(1993), Ma and Clarke(1995), Poole et al(1996), Saha et al(2001), Elmustafa and Stone(2002) and the microbending experiments on very thin nickel beams, Stolken and Evans(1998), Shrotiya et al(2003) and Nanoindentation experiments on microelectronics packaging solder joints by Gomez and Basaran (2006) An additional source of development for strain gradient theories is identified in problems where material instabilities develop as a result, once again, of deformation-localization. For instance, in the simple tension test deformation-localization manifest itself in the so-called necking phenomenon. Similar problems correspond to metallic materials under impact, concrete and rock under tension or unconfined compression and shear of heavily consolidated soils. In these problems, as plastic localization takes place the governing differential equations change type from elliptic to hyperbolic, Mühlhaus and Aifantis(1991). As a result, there is a breakdown in the numerical solution of the problem. In the case of the finite element method, the response becomes highly mesh dependent and it is not possible to preserve a monotonic convergence rate upon mesh refinement. In order to regularize the governing differential equations and eliminate the mesh dependence, a length scale is frequently introduced into the formulation. In this case the length scale is apparently imposed by a numerical requirement although it has often been given a physical meaning tied to the width of shear bands that appear once localization takes place. A similar regularization technique is often achieved via introduction of an artificial viscoplastic effect on the constitutive description and where the rate independent behavior is
reached as the limit of a consistent rate dependent framework, Simo and Hughes(1997). Once the gradients of strain are considered additional kinematic variables and stress definitions appear into the formulation. They may appear directly like in the class of theories promulgated by Fleck and Hutchinson(1993), Fleck et al(1993), Fleck and Hutchinson(1997) or indirectly like in the theories by Aifantis(1992) and De Borst and Mühlhaus(1992). In the former case the gradients of strain appear directly into the governing differential equations giving rise to additional boundary conditions. In the latter case the gradients of strain enter into the yield surface definition, giving rise to a differential equation in the consistency parameter and the consistency condition is now satisfied only in a weak sense. In either case the presence of the gradients of strain demand for higher orders of continuity in the shape functions used in a finite element implementation in order to preserve the monotonic convergency rate of the solution upon mesh refinement. This means that so called complete $C^1$ element shape functions are needed. First order continuous elements are computationally expensive, their implementation into existing finite element platforms is complicated and in general they are not readily available. Alternatively, one can use $C^0$ continuous elements and impose the existing kinematic constraints in some average sense, for instance making use of a Lagrange multipliers technique or equivalently a penalty function/reduced integration scheme. In the present article we focus in such a problem. We use a so-called Cosserat couple stress based strain gradient plasticity theory and discuss its implementation in the commercial finite element code ABAQUS. The implementation is in the form of a user element subroutine UEL. In the first part of the paper we present the Cosserat theory and discuss about the new kinematic variables that allow the inclusion of gradients of strain in the form of curvatures. In the following section we focus in the different alternatives for the finite element implementation of the theory. The discussion is
presented in terms of a total potential energy functional where the kinematic constrains are treated rigorously or enforced in some average sense. Once the framework is implemented into the finite element method we append a nonlinear material response capability after generalizing the theory to the case of rate independent plastic behavior. The second part of the paper discusses the flow theory equations and the needed integration algorithm. In the last part we present simulations of the Stolken and Evans (1998) microbending experiment to show the validity of the implementation.

2.0 The Cosserat Couple Stress Theory.

The description that follows refers to a general solid occupying a volume $\Omega$ and bounded by a surface $\partial\Omega$ defined by an outward normal vector $n$. Throughout we will refer to volume differentials as $d\Omega$ and surface area differentials as $d\Gamma$. There are different interpretations of a Cosserat continuum. In the Cosserat and Cosserat (1909) couple stress theory a differential material element admits not only normal and shear stresses but also couple stress components. For linear elastic behavior the usual stress components are functions of the strains and the couple stresses are functions of the strain gradients. Two distinct theories are identified in the work of the Cosserats as described by Aero and Kuvshinski (1960), Mindlin (1964), De Borst (1993) and Shu and Fleck (1999). First, there is a reduced couple stress theory with the kinematic quantities being the displacement $u_i$ and an associated material rotation $\theta_i$ tied to the displacements by the kinematic constraint given in Eq. 1 and with the definitions of strain and curvatures expressed in Eq. 2 and Eq. 3,

$$\theta_i = \frac{1}{2} e_{ik} u_{k,i} \quad (1)$$

$$\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \quad (2)$$
\( \chi_{ij} = \theta_{i,j} \) 

Within this theory the continuum is assumed to possess bending stiffness allowing for the introduction of additional stress measures in the form of moments per unit area. The presence of the couple stresses renders the Cauchy stress tensor asymmetric, however only the symmetric component generates work upon deformation. Second, there is a general couple stress theory in terms of a microrotation \( \omega_i \) which is regarded as an independent kinematic variable. The rotation and microrotation are related by a relative rotation tensor \( \alpha_{ij} = e_{ik} \omega_k - e_{ik} \theta_k \). For the particular choice of \( \omega_k = \theta_k \) the general couple stress theory reduces to the more restrictive reduced couple stress theory. This general theory assumes that within the material element there is also embedded a micro-volume giving rise to the microrotation \( \omega_i \). Different theories have been postulated depending on the deformation properties assumed for the micro-volume, see Mindlin and Tiersten(1962), Toupin(1962) and Mindlin(1964, 1965) for a review of the different approaches. For instance Figure 2 shows the deformation state in a material point including the micro-volume for the case of pure shear in the solid proposed by Mindlin(1964). Figure 3 shows the particular case of reduced couple stress theory where the micro-volume is assumed rigid.

Equilibrium of the differential element shown in Figure 3 (which is valid for both theories), after neglecting body forces and body couples yields

\[
\sigma_{ji,j} + \tau_{ji,j} = 0
\]  

(4)

\[
\tau_{jk} + \frac{1}{2} e_{jk} m_{pi,p} = 0
\]  

(5)
where $\sigma_{ij}$ and $\tau_{ij}$ are the symmetric and anti-symmetric components of the Cauchy stress tensor respectively, and $m_{ij}$ is the couple stress tensor. Surface stress tractions and surface couple stress tractions are given by

$$t_i = (\sigma_{ij} + \tau_{ij})n_j \quad (6)$$

$$q_i = m_{ij}n_j \quad (7)$$

where $t_i$ is the traction vector, $q_i$ is the couple tractions vector and $n_j$ is a surface normal vector.

In order to place the framework within a more general context it follows that for the case of linear elastic material behavior, the Cosserat couple stress solid can be obtained after postulating a strain energy density function dependent on strains and curvatures (rotation gradients). Toupin(1962) and Mindlin(1965) extended this theory to include also stretch gradients.

3.0 Variational statements.

3.1 Total potential energy functional.

The following elastic constitutive relationships can be written for the symmetric Cauchy stress tensor, the asymmetric stress tensor and the couple stress tensor.

$$\sigma_{ij} = C_{ijkl} e_{kl}$$

$$\tau_{ij} = D_{ijkl} e_{kl} \quad (8)$$

$$\ell^{-1} m_{ij} = D_{ijkl} \chi_{kl}$$

where $C_{ijkl}$ is a constant constitutive tensor relating strains to Cauchy stresses, $D_{ijkl}$ is a constant constitutive tensor relating curvatures to Couple stresses and $D_{ijkl}$ is a constant constitutive tensor relating relative rotations to the anti-symmetric component of the Cauchy stress tensor.

The total potential energy functional $\Pi$ for the general elastic couple stress solid can be written
considering separately the contributions from the symmetric and anti-symmetric stress tensors and the couple stress tensor as given by Eq.9a,

\[
\Pi(u_i, \omega_i) = \frac{1}{2} \int_{\Omega} C_{ijkl} \varepsilon_{ij}(u_i) \varepsilon_{kl}(u_i) d\Omega + \frac{1}{2} \int_{\Omega} D_{ijkl} \chi_{ij}(\omega_i) \chi_{kl}(\omega_i) d\Omega + \frac{1}{2} \int_{\Omega} \bar{D}_{ijkl} \alpha_{ij}(u_i, \omega_i) \alpha_{kl}(u_i, \omega_i) d\Omega - \int_{\partial\Omega} t_i u_i d\Gamma - \int_{\partial\Omega} q_i \omega_i d\Gamma
\]

(9a)

Note that in the functional representation of Eq.9a the translational and rotational degrees of freedom are regarded as independent kinematic quantities. For the discussion that follows it is useful to write Eq.9a in the equivalent form of Eq.9b

\[
\Pi(u, \omega) = F(u) + G(\omega) + H(u, \omega)
\]

(9b)

where

\[
F(u) = \frac{1}{2} a(u, u) - f(u)
\]

\[
G(\omega) = \frac{1}{2} b(\omega, \omega) - g(\omega)
\]

\[
H(u, \omega) = \frac{1}{2} \int_{\Omega} \bar{D}_{ijkl} \alpha_{ij}(u_i, \omega_i) \alpha_{kl}(u_i, \omega_i) d\Omega
\]

with \(a(u, u) \equiv \int_{\Omega} C_{ijkl} \varepsilon_{ij}(u_i) \varepsilon_{kl}(u_i) d\Omega\) and \(b(\omega, \omega) \equiv \int_{\Omega} D_{ijkl} \chi_{ij}(\omega_i) \chi_{kl}(\omega_i) d\Omega\) being symmetric bilinear forms and \(f(u), g(\omega)\) corresponding to the boundary terms in Eq.9a. For the particular case of the more restricted reduced couple stress theory where the stress tensor is symmetric and satisfies the constraint in Eq.1, \(\alpha_{ij}\) (or equivalently \(H(u, \omega)\)) vanishes and equation Eq.9a reduces to Eq.10a,

\[
\Pi(u_i) = \frac{1}{2} \int_{\Omega} C_{ijkl} \varepsilon_{ij}(u_i) \varepsilon_{kl}(u_i) d\Omega + \frac{1}{2} \int_{\Omega} D_{ijkl} \chi_{ij}(u_i) \chi_{kl}(u_i) d\Omega - \int_{\partial\Omega} t_i u_i d\Gamma - \int_{\partial\Omega} q_i \omega_i (u_i) d\Gamma
\]

(10a)
where now the strain energy contribution from the curvatures becomes a function of the translational degrees of freedom only. Note that Eq.10a is analogous to the total potential energy functional for the particular case of the so-called Timoshenko beam theory. Using the introduced alternative notation of Eq.10a the total potential energy functional in a reduced theory coupled stress solid can be written as

\[ \Pi(u) = F(u) + G'(u) \]  

(10b)

where \( G'(u) = \frac{1}{2} b'(u,u) - g'(u) \) with \( b'(u,u) = \int_D g^{ijkl} \chi_{ij} u_{ijkl} u_j d\Omega \) and \( g'(u) \) corresponding to the boundary part associated to the rotation. The prime superscript notation \((\cdot)'\) has been introduced in order to clarify the fact that in the reduced theory the curvatures are kinematically constrained to the displacements. Eq.9 and Eq.10 are the basis for the finite element implementation of the theories described in section 2. Observation of Eq.9b and Eq.10b suggests that the reduced theory solid can also be formulated in terms of independent rotational components with the constraint to the translational components considered as the limit when the term \( H(u,\omega) \rightarrow 0 \) in the general coupled stress solid. This implies that Eq.10a is written in terms of additional independent rotational components but with the additional requirement imposed by the constraint between translational and rotational displacements. In this case the reduced theory can also be described by the functional \( \hat{\Pi}(u,\omega) \) that follows,

\[ \hat{\Pi}(u,\omega) = \int_\Omega [\frac{1}{2} C_{ijkl} \epsilon_{ij}(u_i) \epsilon_{kl}(u_j) d\Omega + \frac{1}{2} D_{ijkl} \chi_{ij}(u_i) \chi_{kl}(u_j) d\Omega - I_i u_i d\Gamma - \int_{\partial\Omega} q_i \omega_i d\Gamma] \]  

(10c)
Comparison of Eq.9b and 10c reveals that $\Pi(u, \omega) \to \tilde{\Pi}(u, \omega)$ when $H(u, \omega) \to 0$. This means that the general couple stress theory approaches the reduced couple stress theory in the limit of vanishing relative rotation $\alpha_{ij}$.

3.2 General variational problem.

In order to progressively show the alternatives for the finite element implementation of the Cosserat couple stress theories (both general and reduced), it is useful to place the formulation within the context of a general variational problem without making explicit reference to any particular theory. The notation to be used herein is standard within the theory of variational calculus and in this particular presentation it corresponds to the one in Oden and Kikuchi(1982). Consider a general solid as the one in Figure 1 occupying a volume $\Omega$, bounded by a smooth surface $\partial \Omega$ and satisfying given kinematic constraints. We define $\tilde{V}$ as a space of admissible functions such that $\Pi : \tilde{V} \to \mathbb{R}$ is a functional defined on $\tilde{V}$. Since interest in this work is on constrained problems, it is then necessary to introduce a subset where the constraint is to be satisfied. For that purpose we designate $\tilde{K}$ as a non-empty subset of $\tilde{V}$ defining the constraint set and with elements specified by $\tilde{K} = \{ \tilde{v} \in \tilde{V} \mid B_c \tilde{v} = \tilde{g}_0 \}$ where $B_c$ is a constraint operator such that $B_c : \tilde{V} \to \tilde{T}$, and $\tilde{g}_0$ is prescribed data defined in some given range $\tilde{T}$. With the use of the above definitions the variational problem that we are interested in, is described by Eq.11 or equivalently by Eq.12 and generally posed as follows. Find $\tilde{u} \in \tilde{K}$ (the constraint set) such that, for any $\tilde{v}$ in $\tilde{K}$, $\Pi$ assumes its minimum value at $\tilde{u}$. This is equivalently written as

$$\tilde{u} \in \tilde{K} : \Pi(\tilde{u}) \leq \Pi(\tilde{v}) \quad \forall \, \tilde{v} \in \tilde{K}$$

The minimizer $\tilde{u}$ of $\Pi$ is then characterized as the solution of the following variational equation, Brezzi and Fortin(1991),
\( \tilde{u} \in \tilde{K} : \langle \mathcal{D} \Pi(\tilde{u}), \tilde{v} \rangle = 0 \quad \forall \tilde{v} \in \tilde{K} \) \hfill (12)

from which Euler equations can be derived. In the variational Eq.12 and from this point on, \( \tilde{u} \) will represent the minimizer function (e.g., the solution) and \( \tilde{v} \) will denote a variation of \( \tilde{u} \).

With this general approach and using the introduced notation corresponding to a general constrained variational problem, we will further extend these ideas to the cases of general and reduced couple stress theories using pure displacement, mixed and penalty/method formulations as suggested by the functionals \( \Pi(u, \omega), \Pi\bar{u}(u) \) and \( \Pi\tilde{u}(u, \omega) \).

3.3 Variational problem 1: General couple stress theory

Consider the case of the general couple stress solid where there is no constraint set but instead there is a space of admissible functions equipped with degrees of freedom having not only translational but also rotational components. We may think of the rotational components as independent elements belonging to the space \( \tilde{Q} \) or alternatively we can define displacements and rotations as elements of the single space \( \tilde{V} \times \tilde{Q} \). In any case the product \( \tilde{V} \times \tilde{Q} \) results in a third space with elements being all the ordered pairs of the form \((\tilde{u}, \tilde{\omega}) \in \tilde{V} \times \tilde{Q}\). In terms of the introduced notation this is equivalent to the following variational problem; find \((\tilde{u}, \tilde{\omega}) \in \tilde{V} \times \tilde{Q}\) such that for any \((\tilde{v}, \tilde{\phi}) \in \tilde{V} \times \tilde{Q}\), \( \Pi \) assumes its minimum value at \((\tilde{u}, \tilde{\omega}) \) where \( \tilde{V} \times \tilde{Q} \) is now the corresponding space of admissible functions. In abstract compact notation this is written as

\[(\tilde{u}, \tilde{\omega}) \in \tilde{V} \times \tilde{Q} : \Pi(\tilde{u}, \tilde{\omega}) \leq \Pi(\tilde{v}, \tilde{\phi}) \quad \forall (\tilde{v}, \tilde{\phi}) \in \tilde{V} \times \tilde{Q} \] \hfill (13)

The minimizer \((\tilde{u}, \tilde{\omega}) \) of \( \Pi \) is characterized by the variational equation

\[ \langle \mathcal{D} \Pi(\tilde{u}, \tilde{\omega}), (\tilde{v}, \tilde{\phi}) \rangle = 0 \quad \forall (\tilde{v}, \tilde{\phi}) \in \tilde{V} \times \tilde{Q} \] \hfill (14)
where \( \hat{D} \) is a differential operator such \( \langle \hat{D}(\tilde{u}, \tilde{\omega}), (\tilde{v}, \tilde{\phi}) \rangle \) denotes the first variation in \( \Pi \) at \((\tilde{u}, \tilde{\omega})\) in direction \((\tilde{v}, \tilde{\phi})\). Identifying \( \Pi \) with the one in Eq.9a leads to the following generalized form of the principle of virtual displacements for the general couple stress theory,

\[
\int \sigma_{ij} \varepsilon_{ij}(v_i) d\Omega + \int m_{ij} X_{ij}(\phi) d\Omega + \int \tau_{ij} \alpha_{ij}(v_i, \phi_i) d\Omega - \int t_i v_i d\Gamma - \int q_i \phi_i d\Gamma = 0 \tag{15}
\]

where \( v_i \) and \( \phi_i \) denote virtual variables. As stems from the principle of virtual displacements in Eq.15, \((\tilde{u}, \tilde{\omega})\) are regarded as independent kinematic functions. This is in fact consistent with the definition given in section 2.0 of the general couple stress theory and the resulting variational problem is thus unconstrained without any need for the subset \( K \) as in section 3.1. This is precisely the formulation used by De Borst(1993) within the context of strain-softening media.

### 3.4 Variational problem 2: Reduced couple stress theory

For the case of the reduced couple stress theory the variational problem can be described as follows; find \( \tilde{u} \in \tilde{V} \) such that for any \( \tilde{v} \in \tilde{V} \) \( \tilde{\Pi} \) assumes its minimum value at \( \tilde{u} \) or equivalently

\[
\tilde{u} \in \tilde{V} : \Pi(\tilde{u}) \leq \Pi(\tilde{v}) \quad \forall \tilde{v} \in \tilde{V} \tag{16}
\]

The minimizer \( \tilde{u} \) of \( \Pi \) is characterized by the following variational equation

\[
\langle \hat{D}(\tilde{u}), \tilde{v} \rangle = 0 \quad \forall \tilde{v} \in \tilde{V} \tag{17}
\]

Identifying \( \Pi \) with the one in Eq.10a leads to the following form of the principle of virtual displacements for a reduced couple stress theory,
In contrast to the case defined in Eq.15, the present principle of virtual displacements shows that the only kinematic variable is $\bar{u}$ which at the same time completely defines the strains $\epsilon_{ij}$ and the curvatures $\chi_{ij}$. This is consistent with the definition given in section 2.0 of the reduced couple stress theory. Notice that in this formulation there are no independent rotational components for the admissible displacements and there is no constraint to be enforced. However the displacement functions need to be $C^1$ continuous in a finite element implementation. In this sense the above problem can be characterized as an unconstrained variational problem without the need to define a subset $K$ as in section 3.2. Xia and Hutchinson(1996) used Eq.18 with degrees of freedom being the displacements and displacements derivatives. In that particular implementation the kinematic constraint between displacements and rotations was not enforced therefore leading to expected pathological mesh dependencies. Wei and Hutchinson(1997) used Eq.18 with degrees of freedom being only the translational components of displacement and with $C^0$ continuous interpolation functions in a clear violation of the continuity conditions demanded by the finite element method. Begley and Hutchingsn(1998) used the higher order equivalent of Eq.18 where displacement and displacement derivatives were considered as nodal degrees of freedom with no enforcement of the kinematic constrain. This approach guarantees $C^1$ continuity only at the nodes and not along the complete inter-element boundaries analogously to the elements by Xia and Hutchinson(1996).

3.5 Variational problem 3 (Reduced couple stress theory)-Mixed variational principle

The reduced theory problem can also be formulated as a constrained variational problem in terms of Eq.9b where $(\bar{u}, \bar{\omega}) \in \bar{V} \times \bar{Q}$ are treated as functions subject to the constraint

$$\int_{\Omega} \sigma_{ij} \epsilon_{ij}(v_i) d\Omega + \int_{\Omega} m_{ij} \chi_{ij}(v_i) d\Omega - \int_{\partial \Omega} t_i v_i d\Gamma - \int_{\partial \Omega} q_i \phi_i(v_i) d\Gamma = 0 \quad (18)$$
established by Eq.1. This approach requires the definition of a constraint set \( \tilde{K} \) with elements specified by

\[
K = \left\{(\tilde{v}, \tilde{\phi}) \in \tilde{V} \times \tilde{Q} \mid B(\tilde{v}, \tilde{\phi}) = g_0 \right\}
\]

where \( B \) is the differential constraint operator and \( g_0 \) is the constraint data with range \( \mathfrak{R}(B) \subset \tilde{T} \). The constraint problem can now be stated as; find \((\tilde{u}, \tilde{\omega}) \in \tilde{K}\) such that, for any \((\tilde{v}, \tilde{\phi}) \in \tilde{K}, \hat{\Pi}\) assumes its minimum value at \((\tilde{u}, \tilde{\omega})\).

This is equivalent to the statement in Eq.19,

\[
(u, \omega) \in \tilde{K} : \hat{\Pi}(u, \omega) \leq \hat{\Pi}(\tilde{v}, \tilde{\phi}) \quad \forall (\tilde{v}, \tilde{\phi}) \in \tilde{K} \quad (19)
\]

The Lagrange multipliers approach to the problem posed by Eq.19 consists of searching for saddle (equilibrium) points \((\tilde{u}, \tilde{\omega}, \tilde{\tau}) \in \tilde{V} \times \tilde{Q} \times \tilde{T}'\) of the following Lagrange functional

\[
L : \tilde{V} \times \tilde{Q} \times \tilde{T}' \to \mathbb{R} ; L(\tilde{v}, \tilde{\phi}, \tilde{\rho}) = \hat{\Pi}(\tilde{v}, \tilde{\phi}) - \left[ \tilde{\rho}, B(\tilde{v}, \tilde{\phi}) - g_0 \right]
\]

where \( \tilde{T}' \) is the dual space of \( \tilde{T} \) and corresponds to the space of admissible Lagrange multipliers \( \tilde{\tau} \) (asymmetric component of the stress tensor) and \( [\cdot, \cdot] \) denotes duality pairing on \( \tilde{T} \times \tilde{T}' \) (i.e., inner product). In this expression \( \tilde{\rho} \) is related to \( \tilde{\tau} \) in the same form in which \( \tilde{v} \) is related to \( \tilde{u} \).

The unique saddle point of \((\tilde{u}, \tilde{\omega}, \tilde{\tau})\) is then characterized by the following variational equation,

\[
\left\langle \hat{D}\hat{\Pi}(\tilde{u}, \tilde{\omega}), (\tilde{v}, \tilde{\phi}) \right\rangle - \left[ \tilde{\tau}, B(\tilde{v}, \tilde{\phi}) \right] = 0 \quad \forall (\tilde{v}, \tilde{\phi}) \in \tilde{V} \times \tilde{Q} \quad (20a)
\]

\[
[\tilde{\rho}, B(\tilde{u}, \tilde{\omega})] = [\tilde{\tau}, g_0] \quad \forall \tilde{\tau} \in \tilde{T}' \quad (20b)
\]

Identifying \( \hat{\Pi}(\tilde{u}, \tilde{\omega}) \) with the one in Eq.10c leads to the following principle of virtual displacements with the kinematic constraint being imposed in a weak sense in Eq.21b,

\[
\int_{\Omega} \sigma_{y} \varepsilon_{y}(v) d\Omega + \int_{\Omega} m_{y} \chi_{y}(\phi) d\Omega - \int_{\Omega} \tau_{y} \alpha_{y}(v, \phi) d\Omega = \int_{\partial\Omega} t_{y} v_{d} d\Gamma + \int_{\partial\Omega} q_{y} \phi_{d} d\Gamma \quad (21a)
\]
\[ \int \rho \alpha_i \partial_i u \partial_i \alpha_i d\Omega = 0 \]

(21b)

The first two terms on the right hand side of Eq.21a corresponds to virtual work done by the stresses and couple stresses respectively. The third term corresponds to the virtual work done by the asymmetric component of the stress tensor. The unique term in Eq.21b corresponds to the weak enforcement of the constraint condition of vanishing relative rotation. This approach was used by Xia and Hutchinson(1996) as an alternative to solve the mesh dependent problems when using Eq.18 and a pure displacement formulation. The present mixed approach is completely equivalent to the one proposed by Herrman(1983) and has also been used by Shu et al(1999) and Shu and Fleck(1999). Other mixed approaches are proposed in the latter reference where comments are available about the numerical issues of this type of numerical strategy.

3.6 Variational problem 4: Reduced couple stress theory)-Penalty method

The Couple stress reduced theory problem can also be formulated in terms of Eq.9a where \( \tilde{u}, \tilde{\omega} \) are treated as independent functions and the constraint established by Eq.1 is imposed via a penalty parameter approach after letting \( \tau_{ij} = G\alpha_{ij} \) where \( G \) is a so-called penalty number. This approach is derived as a result of regularizing the Lagrangian \( L(\tilde{v}, \tilde{\phi}, \tilde{\rho}) \) in section 3.5 in order to avoid possible losses of uniqueness in the mixed formulation approach. Proceeding with the regularization process results in the following alternative functional,

\[
L_{\beta} : \tilde{V} \times \tilde{Q} \times \tilde{T} \rightarrow \mathbb{R} ; L_{\beta}(\tilde{v}, \tilde{\phi}, \tilde{\rho}) = L(\tilde{v}, \tilde{\phi}, \tilde{\rho}) - \frac{1}{2} \beta \| \tilde{\rho} \|^2.
\]

(22)

where \( \beta \) is a parameter. The Lagrange multipliers approach when applied to Eq.22 yields the following variational equation which is analogous to Eq.20,
\[
\begin{align*}
&D\hat{\Pi}(\bar{u},\bar{\omega}),(\bar{v},\bar{\phi}) - [\bar{r}, B_c(\bar{v},\bar{\phi})] = 0 \quad \forall \ (\bar{v},\bar{\phi}) \in \bar{V} \times \bar{Q} \\
&[\bar{\rho}, B_c(\bar{u},\bar{\omega})] + \beta \left[ \bar{\rho}, j^{-1}(\bar{r}) \right] = [\bar{\rho}, g_0] \quad \forall \ \bar{\rho} \in \bar{T}'
\end{align*}
\] (23a)

(23b)

In Eq.23 the operator \( j \) represents a mapping relating \( \alpha \) and \( \bar{r} \) as \( j : \bar{T}' \to \bar{T} \). Solving Eq.23b for the Lagrange multiplier \( \bar{r} \) and using this result in Eq.23a results in the following equivalent variational equation.

\[
\begin{align*}
&D\hat{\Pi}(\bar{u},\bar{\omega}),(\bar{v},\bar{\phi}) + \frac{1}{\beta} \left[ j_\beta \left( B_c(\bar{u},\bar{\omega}) - g_0, B_c(\bar{v},\bar{\phi}) \right) \right] = 0 \quad \forall \ (\bar{v},\bar{\phi}) \in \bar{V} \times \bar{Q}
\end{align*}
\] (24)

Two aspects need to be pointed out in Eq.24. First, the only involved function is \( (\bar{u},\bar{\omega}) \in \bar{V} \times \bar{Q} \) and the Lagrange multiplier has been eliminated. Second, this equation corresponds to the characterization of the minimizer of the following penalty functional

\[
\hat{\Pi}_\beta(\bar{v},\bar{\phi}) = \hat{\Pi}(\bar{v},\bar{\phi}) + \frac{1}{2\beta} \| B_c(\bar{u},\bar{\omega}) - g_0 \|^2
\] (25)

In terms of Eq.25 the following variational problem results; find \( (\bar{u},\bar{\omega}) \in \bar{V} \times \bar{Q} \) where \( (\bar{u},\bar{\omega}) \) are constrained by Eq.1 and such that for any \( (\bar{v},\bar{\phi}) \in \bar{V} \times \bar{Q} \) the functional \( \hat{\Pi}_\beta(\bar{v},\bar{\phi}) \) has a minimum point at \( (\bar{u},\bar{\omega}) \). This is equivalent to Eq.26,

\[
(\bar{u},\bar{\omega}) \in \bar{V} \times \bar{Q} : \hat{\Pi}_\beta(\bar{u},\bar{\omega}) \leq \hat{\Pi}_\beta(\bar{v},\bar{\phi}) \quad \forall \ (\bar{v},\bar{\phi}) \in \bar{V} \times \bar{Q}
\] (26)

The minimum \( (\bar{v},\bar{\phi}) \) of \( \hat{\Pi}_\beta \) is characterized by the following variational equation

\[
\begin{align*}
&D\hat{\Pi}(\bar{u},\bar{\omega}),(\bar{v},\bar{\phi}) + G \left[ \hat{L}\bar{\omega} - M\bar{u}, \hat{L}\bar{\phi} - M\bar{v} \right] = 0 \quad \forall \ (\bar{v},\bar{\phi}) \in \bar{V} \times \bar{Q}
\end{align*}
\] (27)

or equivalently

\[
a(\bar{v},\bar{u}) + b(\bar{\phi},\bar{\omega}) + G \left[ L\bar{\phi} - M\bar{v}, L\bar{\omega} - M\bar{u} \right] = f(\bar{v}) + g(\bar{\phi})
\] (28)
where $G = 1/\beta$, $B_{ij}(\ddot{u}, \ddot{\omega}) = \ddot{\omega} - M\dot{\omega}$ and $f(\ddot{v})$ and $g(\ddot{\phi})$ have been defined in Eq.9. Eq.28 is equivalent to the following form of the principle of virtual displacements for a reduced couple stress theory

$$
\int_{\Omega} \sigma_{ij} \varepsilon_{ij}(v) d\Omega + \int_{\Omega} m_{ij} \chi_{ij}(\phi) d\Omega + \int_{\Omega} G\alpha_{ij} \alpha_{ij}(v, \phi) d\Omega - \int_{\Gamma} t_{ij} d\Gamma - \int_{\Gamma} q_{ij} d\Gamma = 0
$$

(29)

The principle of virtual displacements in the form of Eq.29 has been used by Shu and Fleck(1997) to implement strain gradient theories within ABAQUS. This principle is suggested by Herrman(1983) although the specific details are not available in the reference.

4.0 Finite element discretization-general notation.

This section describes the discretized versions of the equations derived from the variational problems stated above starting from their corresponding variational equations. In each case we will schematically show a typical finite element with its associated vector of nodal point parameters but without any restriction as to the number of nodes. The parameters may be translational degrees of freedom, translational and rotational degrees of freedom or translational and rotational degrees of freedom with additional Lagrange multipliers. To distinguish between the values of the given parameter at any point within the element and its nodal point value we will use the following notation. For instance if displacement is being considered, the value at any point within the element will be denoted by the vector $u$ and its corresponding nodal point representation will be denoted by $\hat{u}$. In a given element the value of a given parameter at any point within the element is obtained via interpolation from the known nodal point values. Kinematic variables dependent from the basic parameters, as strains and curvatures, are interpolated from the nodal point values using interpolating functions which in generally are derivatives of the basic interpolating functions. We will denote such a function by a subscripted
symbol where the subscript indicates the variable that is being interpolated. For instance we will denote the displacement-curvature interpolation matrix by $B_\chi$ where the curvature at any point within a given element is obtained out of the nodal point displacement vector as $\chi = B_\chi \hat{\mathbf{u}}$ where $\hat{\mathbf{u}}$ may have translational or translational and rotational degrees of freedom.

4.1 Variational problem : General Couple stress theory.

In the case of the general couple stress solid free of the constraint in Eq.1, $\hat{\mathbf{u}}, \hat{\omega}$ are independent degrees of freedom. In the finite element formulation, this fact implies that elements with $C^0$ continuity are enough to satisfy displacement compatibility requirements. The nodal point displacement vector for an n-noded element has the following general form $\hat{\mathbf{u}}_e^T = [u_i \nu_i \omega_1 \ldots \ldots u_n \nu_n \omega_n]$ as shown in Figure 4. To describe the finite element discretization we recall Eq.3-15 for a given general couple stress theory element

\[
\int_{\Omega_\epsilon} \sigma_{ij} \varepsilon_{ij} (\nu_i) d\Omega + \int_{\Omega_\epsilon} m_{ij} \chi_{ij} (\phi_i) d\Omega + \int_{\Omega_\epsilon} \tau_{ij} \chi_{ij} (\nu_i, \phi_i) d\Omega - \int_{\partial\Omega_\epsilon} t_i \nu_i d\Gamma - \int_{\partial\Omega_\epsilon} q_i \phi_i d\Gamma = 0 \tag{30a}
\]

Letting $u_e = N\hat{u}_e$, $\varepsilon_e = B_e \hat{\mathbf{u}}_e$, $\chi_e = B_e \hat{\chi}_e$, $\alpha_e = B_a \hat{\mathbf{u}}_e$ and $\tau_e = N\hat{\tau}_e$ where $\hat{\tau}_e^T = [t_i \nu_i \omega_1 \ldots \ldots t_n \nu_n \omega_n]$ and using the constitutive relationships introduced in Eq.9 we have that Eq.30a can be written in the following matrix form after eliminating the virtual variables

\[
\begin{bmatrix}
\int_{\Omega_\epsilon} B_e^T C B_e d\Omega + \int_{\Omega_\epsilon} B_e^T D B_e d\Omega + \int_{\Omega_\epsilon} B_a^T \tilde{D} B_a d\Omega
\end{bmatrix} \hat{\mathbf{u}}_e = \int_{\partial\Omega_\epsilon} N^T \tilde{t} d\Gamma \tag{30b}
\]

or equivalently using stiffness matrices

\[
\begin{bmatrix}
K_e^\varepsilon + K_e^\chi + K_e^\alpha
\end{bmatrix} \hat{\mathbf{u}}_e = \hat{\mathbf{F}}_e \tag{30c}
\]
4.2 Variational problem 2: Reduced Couple stress theory-Pure displacement formulation.

In the case of the reduced couple stress theory $\bar{u}, \bar{\omega}$ are related through the constraint relationship in Eq.1 and are not independent of each other. In terms of a formulation based on translational degrees of freedom only, this constraint implies that elements with $C^1$ continuity are needed to satisfy displacement compatibility requirements. Assuming that such an element is readily available, the nodal point displacement vector for an n-noded element has the following general form $\hat{u}_e^T = [u_1, v_1, \ldots, u_n, v_n]$ as shown in Figure 5.

To describe the finite element discretization we recall Eq.18 for a given reduced couple stress theory element

$$\int_\Omega \sigma_{ij} \epsilon_{ij}(v_i) d\Omega + \int_\Omega m_{ij} \chi_{ij}(v_i) d\Omega - \int_{\partial\Omega} t_i v_i d\Gamma - \int_{\partial\Omega} q_i \phi_i(v_i) d\Gamma = 0 \quad (31a)$$

Letting $u_e = N\hat{u}_e$, $\epsilon_e = B_e \hat{u}_e$, $\chi_e = B_e \hat{\epsilon}_e$, and $q = N_q^T \hat{t}$ and the constitutive relationships introduced in Eq.9 we have that Eq.31a can be written in the following matrix form after eliminating the virtual variables,

$$\begin{bmatrix} \int_{\Omega_e} B_e^T C B_e d\Omega + \int_{\Omega_e} B_e^T D B_e d\Omega \end{bmatrix} \hat{u}_e = \int_{\partial\Omega_e} (N_e^T + N_q^T) \hat{t} d\Gamma \quad (31b)$$

or equivalently using stiffness matrices

$$\begin{bmatrix} K_e^e + K_e^q \end{bmatrix} \hat{u}_e = \hat{F}_e \quad (31c)$$
In this case the matrix $N_q^T$ interpolates rotations from translational nodal point values and it contains derivatives of the displacement shape functions.

4.3 Variational problem 3: -Reduced Couple stress theory-Lagrange multipliers formulation.

In the case of the reduced couple stress theory $\tilde{u}, \tilde{\omega}$ are related through the constraint relationship in Eq.1 and are not independent of each other. Here we present a formulation based on translational and rotational degrees of freedom with the kinematic constraint enforced via Lagrange multipliers approach. In this case the problem can be treated in terms of two vectors of nodal point parameters. The displacement degrees of freedom vector $\tilde{u}^T = [u_1, v_1, \omega_1, ..., u_n, v_n, \omega_n]$ and an additional nodal point vector of Lagrange multipliers $\tilde{\tau}^T = [\tau_1, ..., \tau_n]$ as shown in Figure 6. To describe the finite element discretization we recall Eq.21 for a given reduced couple stress theory element enforcing the kinematic constraint via Lagrange multipliers

$$\int_\Omega \sigma \varepsilon \omega \phi d\Omega + \int_\Omega m \chi \omega \phi d\Omega - \int_\Gamma \alpha \tau \phi d\Gamma = \int_\Gamma \nu \phi d\Gamma + \int_\Gamma q \phi d\Gamma$$

$$\int_\Omega \rho \alpha \omega \phi d\Omega = 0$$

Using $u_e = N_u \hat{u}_e$, $\tau_e = N_\tau \hat{\tau}_e$, $\varepsilon_e = B_\varepsilon \hat{u}_e$, $\chi_e = B_\chi \hat{u}_e$, $\alpha_e = B_\alpha \hat{u}_e$, $\tau_e = N_\tau \hat{\tau}_e$ where $\hat{\tau}_e = [\hat{\tau}_1, \hat{\tau}_1, \hat{\tau}_n, ..., \hat{\tau}_1, \hat{\tau}_n, \hat{\tau}_n, ..., \hat{\tau}_n]$ and the constitutive relationships introduced in Eq.9 we have that Eq.32 can be written in the following matrix form,

$$\begin{bmatrix} \int_{\Omega_e} B_x^T C B_x d\Omega + \int_{\Omega_e} B_x^T D B_x d\Omega & \int_{\Omega_e} B_x^T N_{\tau} d\Omega \\ \int_{\Omega_e} N_{\tau}^T B_x d\Omega & 0 \end{bmatrix} \begin{bmatrix} \hat{u}_e \\ \hat{\tau}_e \end{bmatrix} = \begin{bmatrix} \int_{\Gamma_e} N^T \tilde{t} d\Gamma \\ 0 \end{bmatrix}$$
or equivalently using stiffness matrices

\[
\begin{bmatrix}
K_{uu} & K_{ur} \\
K_{ru} & 0
\end{bmatrix}
\begin{bmatrix}
\hat{u}_e \\
\hat{\tau}_e
\end{bmatrix} =
\begin{bmatrix}
\hat{F}_e \\
0
\end{bmatrix}
\]  

(32d)

4.4 Variational problem 4: -Reduced Couple stress theory-Penalty based/reduced integration.

The reduced Couple stress theory can be implemented using the penalty function approach described in section 3.6. In this particular case there is no need to use Lagrange multipliers in order to enforce the kinematic constraint and the nodal point displacement vector for an n-noded element takes the following general form \( \hat{u}_e = [u_1, \ldots, u_n] \equiv [\hat{u}_e, \hat{\omega}_e] \) as shown in Figure 7. To describe the finite element discretization we recall Eq.29 for a given reduced couple stress theory element using a penalty based approach

\[
\int_\Omega \sigma_j \varepsilon_j (v) d\Omega + \int_\Omega m_{ij} \chi_j(\phi) d\Omega + \int_\Omega G \alpha_j \alpha_j (v, \phi) d\Omega - \int_{\partial \Omega} t_i d\Gamma - \int_{\partial \Omega} q_i d\Gamma = 0 
\]  

(33a)

Anticipating the implementation of a nonlinear material constitutive behavior it is convenient to express the constitutive relationships as

\[
\Sigma = ME \equiv \begin{bmatrix}
C & 0 \\
0 & D
\end{bmatrix} \begin{bmatrix}
\varepsilon \\
\chi
\end{bmatrix} 
\]  

(33b)

where \( \sigma_j \) and \( 1^{-1}m_{ij} \) and \( \varepsilon_j \) and \( 1\chi_j \) have been collapsed into \( \Sigma \) and \( E \) respectively, and the constitutive tensors \( C \) and \( D \) into \( M \). In similar form, \( t_i \) and \( q_i \) are collapsed into a single tractions vector \( \bar{t} \) and the general displacement vector including translational and rotational degrees of freedom into a single displacement vector \( \bar{u} \). Using this notation Eq.33a is re-expressed in matrix form as

\[
\int_\Omega \delta \bar{E}^T \Sigma dV + \int_\Omega G_{ij} \delta \alpha_i \alpha_j dV - \int_{\partial \Omega} \delta \bar{u}^T \bar{t} dS = 0 
\]  

(33c)
The displacements in a \( n \)-noded element are discretized in standard form,

\[
\begin{bmatrix}
\hat{u}_e \\
\hat{\omega}_e
\end{bmatrix} = \begin{bmatrix}
N^1, 0, N^2, \ldots, N^N, 0 \\
0, N_\omega^1, 0, \ldots, 0, N_\omega^N
\end{bmatrix} \begin{bmatrix}
\hat{u}_e \\
\hat{\omega}_e
\end{bmatrix} \equiv \bar{u}_e = N\hat{u}_e
\] (33d)

where \( \hat{u}_e \) and \( \hat{\omega}_e \) are translational and rotational nodal degrees of freedom as shown in Figure 7 and \( N \) are displacement interpolation functions. In similar form the strain can be obtained by introducing a generalized strain-displacement operator

\[
B_e = \begin{bmatrix}
B_\varepsilon \\
B_\omega
\end{bmatrix}
= \begin{bmatrix}
\frac{\partial}{\partial x} & 0 & 0 \\
0 & \frac{\partial}{\partial y} & 0 \\
0 & 0 & 0 \\
\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\
0 & 0 & \ell \frac{\partial}{\partial x} \\
0 & 0 & \ell \frac{\partial}{\partial y}
\end{bmatrix}
\] (34)

then \( E = B_e\hat{u}_e \). Similarly the constraint operator \( B_\alpha \) is introduced and such \( \alpha = B_\alpha\hat{u}_e \) and

\[
B_\alpha = \frac{1}{2} \begin{bmatrix}
\frac{\partial}{\partial y} & -\frac{\partial}{\partial x} & N
\end{bmatrix}
\] (35)

Using Eq.33c together with Eq.34 and Eq.35 leads to the following finite element approximation for a given element

\[
\int_{\bar{\Omega}_e} B_e^T MB_e dV + \int_{\bar{\Omega}_e} B_e^T B_\alpha dV \hat{u}_e = \int_{\bar{\Omega}_e} N^T \bar{\tau} dS + \int_{\bar{\Omega}_e} B_\alpha^T \bar{q} dS
\] (36)

where \( \bar{q} \) represents the contribution from the surface tractions corresponding to the asymmetric stress component. Making explicit the contribution from the translational and rotational degrees of freedom equation Eq.36 can be written as

\[
\begin{bmatrix}
K_{uu} & 0 \\
0 & K_{\omega\omega}
\end{bmatrix} \begin{bmatrix}
\hat{u}_e \\
\hat{\omega}_e
\end{bmatrix} + \begin{bmatrix}
K_{uu} & K_{\omega\omega} \\
K_{\omega\omega} & K_{\omega\omega}
\end{bmatrix} \begin{bmatrix}
\hat{u}_e \\
\hat{\omega}_e
\end{bmatrix} = \begin{bmatrix}
f_u \\
f_{\omega}
\end{bmatrix}
\] (37)
In Eq.37 the second matrix corresponds to the contribution from the penalty terms. The following limit cases should be identified in Eq.37. As $G_a \to \infty$ the formulation approaches the reduced couple stress theory. If $\ell \to 0$ and $G_a \to \infty$ then the formulation approaches classical theory. Here classical theory means a solid with translational and rotational degrees of freedom with the kinematic constraint Eq.1 enforced by the penalty term and with no contribution from the couple stress terms. Notice that if in Eq.37 both $\ell \to 0$ and $G_a \to 0$ simultaneously, then the constraint established in Eq.3-1 is violated and the stiffness matrix becomes singular. In Eq.36 or its equivalent Eq.37 the first term in the left is fully integrated whereas the second term depending on the penalty number $G_a$ is integrated using a reduced scheme.

### 4.5 Implementation in ABAQUS-Newton-Raphson iteration.

The formulation described in section 4.4 can be implemented in ABAQUS in a straightforward. This is opposed to the formulations based on the Lagrange multipliers approach where the stiffness matrix has to be manipulated at the global level before the final equations can be solved. Anticipating non-linear material behavior the general algorithm corresponding to the user element subroutine UEL is described in Table 1. In the next section we introduce the flow theory equations for the theory together with a proposed integration algorithm. The integration algorithm corresponds to the subroutine UMAT in Table 1. In the case of nonlinear material behavior the principle of virtual displacements given by Eq.33 or equivalently by Eq.36 still remains valid, and the problem is then posed as follows. If an external load $R_{\text{ext}}$ which is independent of the deformations is applied over some time period (real or mathematical) $[0,T]$ then the basic problem becomes to find the solution to Eq.37 throughout the history of loading application. Since the problem is history dependent the solution must be developed by a series
of small increments. The most common solution scheme is the Newton-Raphson method since it exhibits good convergence characteristics as compared to alternative methods. The Newton iteration algorithm within the present context can be understood as follows. Let

\[ \vec{F} = \int_B \bar{b}_k \bar{\Sigma} dV + \int_B \bar{b}_m \tau dV, \]

then introducing the pseudo-time variable, Eq.37 can be replaced by the following equilibrium statement,

\[ \dot{\vec{R}}_\text{ext} - \dot{\vec{F}} = 0 \]  \hspace{1cm} (38)

where \( \dot{\vec{R}}_\text{ext} \) are the externally applied nodal point forces and \( \dot{\vec{F}} \) are the nodal point forces corresponding to element stresses at time \( t \). Assuming that the solution is known at time \( t \), then the solution at time \( t + \Delta t \) must satisfy

\[ \dot{\vec{R}}_{\text{ext}}^{t+\Delta t} - \dot{\vec{F}}^{t+\Delta t} = 0 \]  \hspace{1cm} (39)

where \( \dot{\vec{R}}_{\text{ext}}^{t+\Delta t} \) is known and

\[ \dot{\vec{F}}^{t+\Delta t} = \dot{\vec{F}}^t + \Delta \vec{F} \]  \hspace{1cm} (40)

with \( \Delta \vec{F} \) being an increment in nodal point forces corresponding to element stresses. If the incremental process is applied by means of small increments, then the following approximation is valid

\[ \Delta \vec{F} \cong \dot{\vec{U}}^t K \Delta \vec{U} \]  \hspace{1cm} (41)

where \( \dot{\vec{K}} = \frac{\partial \dot{\vec{F}}}{\partial \dot{\vec{U}}} \) is the tangent stiffness matrix. Substitution of Eq.41 into Eq.40 and the result into Eq.39 yields,

\[ \dot{\vec{K}} \Delta \vec{U} = \dot{\vec{R}}_{\text{ext}}^{t+\Delta t} - \dot{\vec{F}} \]  \hspace{1cm} (42a)

and then

\[ \dot{\vec{R}}_{\text{ext}}^{t+\Delta t} = \dot{\vec{U}}^t + \Delta \vec{U} \]  \hspace{1cm} (42b)
The solution scheme given by Eq.42 has to be performed iteratively since the approximation given in Eq.41 is used. The full Newton-Raphson algorithm is shown in Box 1. In general terms, the iteration process starts from assumed initial conditions, generally the converged solution from the previous step or in some cases from some extrapolated value (by default in ABAQUS). The iterations are continued until convergence criteria based on displacements, forces or energy are satisfied. A key point in the solution procedure just described and detailed in Table 1 is the updating of the stress and material constitutive tensors once the displacement increment is known. In the case of classical theories efficient and robust integration schemes have been developed and implemented into finite element codes. In the case of gradient enhanced theories this integration algorithm or updating process represents a major challenge.

5.0 Cosserat couple stress based strain gradient plasticity theory.

The continuum model described in the previous section is extended here to incorporate gradient effects via a Cosserat medium. This implies the generalization of the stress space to include couple stresses. In this way the yield surface is considered a sphere in an extended stress space where the classical theory solid represents just a subset of the more general theory. The gradient effects are then introduced by enhancing the definition of equivalent plastic strain with the addition of an equivalent plastic curvature. This approach leads to a straightforward extension of the flow theory description used in a classical solid which allows the treatment of cyclic loading problems as in the case of low cycle fatigue analysis. The formulation that follows presents the flow theory equations for a rate independent material under the assumption of small displacements and small strains. A key feature is the coupling between the Cauchy and Couple stress components. This coupling is not present in the initial elastic material but progressively appears with the development of plastic curvatures as is evident in the continuous
version of the elasto-plastic tangent modulus. In order to integrate the constitutive equations two algorithms are proposed. First a radial return scheme for the rate independent model is presented. This algorithm is outlined with the only purpose in mind of showing the coupling between the stresses and couple stresses evident in the algorithmic version of the elasto-plastic tangent modulus. This integration scheme is not implemented into ABAQUS but rather a return mapping scheme is used. The return mapping scheme can be easily extended to couple stress theory. The integration algorithm is implemented into an ABAQUS user subroutine UMAT which is at the same time called by the UEL subroutine developed in section 4.0. Computational results obtained with the model are compared with the microbending experiments of Stolken and Evans (1998) on thin nickel foils and those of Shrotriya et al (2003) on LIGA nickel foils.

5.1 Rate independent non-linear material behavior.

Recalling the relationship between the symmetric part of the Cauchy stress and the elastic strains and the Couple stresses and the elastic curvatures first presented in Eq.8, they can be written in rate form as follows

\[ \dot{\sigma}_{ij} = C_{ijkl} \dot{\varepsilon}^{el}_{kl} \] (43a)

\[ \ell^{-1} \dot{m}_{ij} = D_{ijkl} \ell \dot{\chi}^{el}_{kl} \] (43b)

where \( D_{ijkl} = \mu \delta_{ik} \delta_{jl} \).

The strains and curvatures are decoupled into elastic and inelastic components which results in,

\[ \dot{\varepsilon}_{ij} = \dot{\varepsilon}^{el}_{ij} + \dot{\varepsilon}^{pl}_{ij} \] (44a)

\[ \ell \dot{\chi}_{ij} = \ell \dot{\chi}^{el}_{ij} + \ell \dot{\chi}^{pl}_{ij} \] (44b)

Considering the following definition of generalized deviatoric stress norm \( \| \Sigma \| \)

\[ \| \Sigma \| = \left[ S_{ij} S_{ij} + \ell^{-1} m_{ij} \ell^{-1} m_{ij} \right]^{1/2} \] (45)
where \( S_{ij} \) is the deviatoric component of \( \sigma_{ij} \) and \( m_{ij} \) is deviatoric in nature. Similarly using the definition of the norm of a second tensor it can be defined for the generalized strain

\[
\|E\| = \left( \varepsilon_{ij} \varepsilon_{ij} + \ell \chi_{ij} \ell \chi_{ij} \right)^{1/2}
\]  

(46)

Next we introduce a yield surface separating the elastic and inelastic domains and in analogous way to the classical Hill plasticity models. In order to consider the Bauschinger effect observed in metals and its alloys a back-stress tensor \( \beta_{ij} \) and a couple back-stress tensors \( \ell^{-1} \eta_{ij} \) defining the displacement of the yield surface in stress space can also be defined. The assumption of the presence of the couple back-stress tensor is motivated theoretically and not based on any experimental evidence. The difference \( f_{ij} = S_{ij} - \beta_{ij} \) between the back-stress and the deviatoric component of the symmetric part of the Cauchy stress tensor is the relative stress tensor \( f_{ij} \). In an analogous form for the couple back-stress, there follows that the relative couple back-stress \( \ell^{-1} \hat{C}_{ij} \) is defined by \( \ell^{-1} \hat{C}_{ij} = \ell^{-1} m_{ij} - \ell^{-1} \eta_{ij} \). The generalized relative stress \( \|\hat{\varepsilon}\| \) can be described in terms of the relative stresses and given by

\[
\|\hat{\varepsilon}\| = \left[ f_{ij} f_{ij} + \ell^{-2} \hat{C}_{ij} \hat{C}_{ij} \right]^{1/2}
\]  

(47)

With Eq.47 at hand a yield surface is introduced. In the classical theory of plasticity the yield surface is defined in terms of a hardening parameter that can be shown to be proportional to the equivalent plastic strain. In the present couple stress based strain gradient plasticity theory this hardening parameter incorporates also the gradient effects via the equivalent plastic curvatures. The generalized yield surface can therefore be expressed as

\[
F(\sigma, \ell^{-1} m, \alpha) = \|\hat{\varepsilon}\| - \sqrt{\frac{2}{3}} K(\alpha)
\]  

(48)
where \( \alpha \) is the generalized hardening parameter and \( K(\alpha) \) represents the radius of the yield surface. In order to complete the description of the flow theory representation of the constitutive model it is necessary to define the flow rules (i.e., evolution equations for the plastic strains and curvatures) and hardening laws (i.e., evolution of the hardening parameter and back-stress components). Here it is assumed that the flow rules obey associative plasticity rules. To this end, it is necessary to define the normal to the yield surface as in Fleck and Hutchinson (1997). Notice that the generalization implied in Eq.48 amounts to consider a more general stress space with normal stresses and couple stresses. Thus the yield surface is still considered as an hypersphere in stress space with normal defined by Eq.49,

\[
\hat{N} = \frac{\partial F}{\partial \Sigma} = [\hat{n}, \hat{\nu}]
\]

where \( \hat{n} = \frac{\partial F}{\partial \sigma} \equiv \frac{f_{ij}}{\xi_{ij}} \) and \( \hat{\nu} = \frac{\partial F}{\partial l^{-1}m} \equiv l^{-1} \hat{C}_{ij} \).

And the flow rules read

\[
\dot{\varepsilon}_{ij}^{pl} = \gamma \frac{\partial F}{\partial \sigma_{ij}} \equiv \gamma f_{ij} \equiv \gamma \hat{n}
\]

(50a)

\[
\ell \dot{\chi}_{ij}^{pl} = \gamma \frac{\partial F}{\partial \ell^{-1} m_{ij}} \equiv \gamma \ell^{-1} \hat{C}_{ij} \equiv \gamma \hat{\nu}
\]

(50b)

In Eq.50 \( \gamma \) is the consistency parameter which is defined from the loading/unloading conditions and is related to the evolution of the generalized equivalent plastic strain defined by the hardening law established in Eq.51

\[
\dot{\alpha} = \frac{2}{\sqrt{3}} \gamma
\]

(51)

The constitutive model is completed with the evolution equations for the back-stresses
\[
\dot{\bar{\gamma}}_j = \frac{2}{3} H^* \gamma \dot{\bar{\gamma}}_j
\]
(52a)

\[
\ell \dot{\bar{\gamma}}_j = \frac{2}{3} H^* \gamma \dot{\bar{\gamma}}_j
\]
(52b)

where \( H^* \) represents a kinematic hardening modulus which may be a linear or a nonlinear function of the hardening parameter \( \alpha \). For instance, the assumption of a constant kinematic hardening modulus leads to the so-called Prager-Ziegler rule, Fung and Tong (2001). Introducing Eq. 50 into the generalized strain norm for the plastic quantities yields

\[
\left\| \dot{E}^{pl} \right\| = \left\| \dot{E}^{pl} + \ell \dot{\dot{E}}^{pl} \right\|^{1/2} = \gamma \left\| \left[ f_j \dot{f}_j + \ell^2 \dot{\bar{\gamma}}_j \dot{\bar{\gamma}}_j + \ell \dot{\dot{\bar{\gamma}}}_j \dot{\bar{\gamma}}_j \right] \right\|^{1/2}
\]

which implies

\[
\left\| \dot{E}^{pl} \right\| = \gamma
\]

Using this result in Eq. 51 and integrating yields

\[
\alpha(t) = \int_0^t \frac{2}{3} \left\| \dot{E}^{pl} (\tau) \right\| d\tau
\]
(53)

Eq. 53 is the generalized version of equivalent plastic strain but with the addition of the gradients of plastic strain. The evolution equations are complemented by the loading/unloading conditions which allow the determination of the consistency parameter. In terms of the yield function defined in Eq. 48 the following loading/unloading condition holds

\[
\gamma \geq 0 \quad F \left( \sigma, \ell^{-1} m, \alpha \right) \leq 0
\]
(54a)

\[
\gamma \geq 0 \quad \gamma F \left( \sigma, \ell^{-1} m, \alpha \right) = 0
\]
(54b)

And the consistency condition

\[
\gamma \dot{F} \left( \sigma, \ell^{-1} m, \alpha \right) = 0
\]
(55)

**Consistency parameter determination.**

Using the definition of the yield function we have
\[
\dot{F} = \frac{\partial F}{\partial \sigma} : \dot{\sigma} + \frac{\partial F}{\partial \ell^{-1} m} : \ell^{-1} \dot{m} + \frac{\partial F}{\partial \varepsilon^p} : \dot{\varepsilon}^p + \frac{\partial F}{\partial \chi^p} : \dot{\chi}^p 
\]

from Hooke’s law Eq.43 together with the flow rules and after collecting terms yields

\[
\dot{F} = [\hat{n} : C : \dot{\varepsilon} + \dot{\varepsilon} : D : \ell \dot{\chi}] - \gamma \left\{ [\hat{n} : C : \dot{n} + \dot{\varepsilon} : D : \dot{\chi}] - \left( \frac{\partial F}{\partial \varepsilon^p} : \dot{n} + \frac{\partial F}{\partial \chi^p} : \dot{\chi} \right) \right\} \quad (56)
\]

and from the consistency condition, there follows for the consistency parameter \( \gamma \) that

\[
\gamma = \frac{\hat{n} : C : \dot{n} + \dot{\varepsilon} : D : \dot{\chi}}{\hat{n} : C : \dot{n} + \dot{\varepsilon} : D : \dot{\chi} - \left( \frac{\partial F}{\partial \varepsilon^p} : \dot{n} + \frac{\partial F}{\partial \chi^p} : \dot{\chi} \right)} \quad (57)
\]

Using

\[
\frac{\partial F}{\partial \varepsilon^p} = -\sqrt{\frac{2}{3}} K \frac{2}{3} \frac{\varepsilon^p}{\alpha} \quad \text{and} \quad \frac{\partial F}{\partial \chi^p} = -\sqrt{\frac{2}{3}} K \frac{2}{3} \frac{\chi^p}{\alpha}
\]

gives

\[
\frac{\partial F}{\partial \varepsilon^p} : \dot{n} = -\sqrt{\frac{2}{3}} K \frac{2}{3} \frac{\varepsilon^p}{\alpha} : S \quad \text{and} \quad \frac{\partial F}{\partial \chi^p} : \dot{\chi} = -\sqrt{\frac{2}{3}} K \frac{2}{3} \frac{\chi^p}{\alpha} : \ell^{-1} m
\]

then the term within brackets in the denominator reduces to

\[
\frac{\partial F}{\partial \varepsilon^p} : \dot{n} + \frac{\partial F}{\partial \chi^p} : \dot{\chi} = -\sqrt{\frac{2}{3}} K \frac{2}{3} \frac{1}{\alpha} \left( \varepsilon^p : S + \ell \chi^p : \ell^{-1} m \right) = -\frac{2}{3} K
\]

after using \( \varepsilon^p : S + \ell \chi^p : \ell^{-1} m = K \alpha \).

Using the previous result the consistency parameter \( \gamma \) reduces to

\[
\gamma = \frac{\hat{n} : C : \dot{n} + \dot{\varepsilon} : D : \dot{\chi}}{\hat{n} : C : \dot{n} + \dot{\varepsilon} : D : \dot{\chi} + \frac{2}{3} K}
\]

Using

\[
\hat{n} : C = 2 \mu \hat{n} \quad \text{and} \quad \hat{n} : C : \dot{n} = 2 \mu \frac{S : S}{\| S \|^2}
\]
\[ \dot{\varepsilon} : D = 2\mu \dot{\varepsilon} \] and \[ \dot{\varepsilon} : C = 2\mu \frac{\ell^{-1} m : \ell^{-1} m}{\|\Sigma\|^2} \]

gives
\[ \gamma = \frac{1}{\|\Sigma\|} \left( S : \mathbb{D} + 1^{-1} : \mathbb{F} \right) \left( 1 + \frac{K^{-1}}{3\mu} \right) \]

Using this result into Hooke’s law yields
\[ \mathbb{D} = C : \mathbb{D} = \frac{2\mu}{K} (\hat{n} \otimes \hat{n}) : \mathbb{D} \]

\[ 1^{-1} \mathbb{K} = D : \mathbb{K} = \frac{2\mu}{K} (\hat{\nu} \otimes \hat{\nu}) : \mathbb{K} \]

Using
\[ C = \kappa \hat{I} \otimes \hat{I} + 2\mu \left( \hat{C} - \frac{1}{3} \hat{I} \otimes \hat{I} \right) \]
and \[ D = 2\mu \hat{C} \]
and after simplifying yields
\[ M^{op} = \begin{bmatrix} \kappa \hat{I} \otimes \hat{I} + 2\mu \left( \hat{C} - \frac{1}{3} \hat{I} \otimes \hat{I} - \frac{\hat{n} \otimes \hat{n}}{K} \right) & -2\mu \frac{\hat{n} \otimes \hat{v}}{K} \\ -2\mu \frac{\hat{n} \otimes \hat{v}}{K} & 2\mu \left( \hat{C} - \frac{\hat{v} \otimes \hat{v}}{K} \right) \end{bmatrix} \]

Alternatively it can be written,
\[ M^{op} = \begin{bmatrix} C - C : \hat{n} \otimes C : \hat{n} & -C : \hat{n} \otimes D : \hat{v} \\ -D : \hat{v} \otimes C : \hat{n} & D - D : \hat{v} \otimes D : \hat{v} \end{bmatrix} \]

The flow theory just described can be written in the following alternative form, which is useful in the numerical treatment of the problem. Using the generalized vector and matrix notation we define for the simple case of isotropic hardening;
\[ \Sigma = \begin{bmatrix} \sigma \\ 1^{-1} m \end{bmatrix}, \quad E = \begin{bmatrix} \varepsilon \\ 1 \chi \end{bmatrix}, \quad M = \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix} \]

then the constitutive model equations can be written as, \( \Sigma = M : E^{el} \). The yield surface and flow rule follow

\[ F(\sigma, \varepsilon^{-1} m, \alpha) = \|\Sigma\| - \sqrt{\frac{2}{3}} K(\alpha) \] (63)

\[ \kappa^e = \gamma \hat{N} \] (64)

then

\[ \kappa^e = \frac{\partial F}{\partial \Sigma} \cdot \kappa^e + \frac{\partial F}{\partial E^{pl}} \cdot \kappa^e \] (65)

Using Hooke’s law and the flow rule results in

\[ \kappa^e = \hat{N} : M : \kappa^e - \gamma \hat{N} : M : \hat{N} + \frac{\partial F}{\partial E^{pl}} \left( \gamma \hat{N} \right) \]

The consistency conditions \( \gamma = 0 \) gives

\[ \gamma = \frac{\hat{N} : M : \kappa^e}{\hat{N} : M : \hat{N} - \frac{\partial F}{\partial E^{pl}} \cdot \hat{N}} \quad \text{or after using} \quad \frac{\partial F}{\partial E^{pl}} \cdot \hat{N} = -\frac{2}{3} K^I \quad \text{we have} \]

\[ \gamma = \frac{\hat{N} : M : \kappa^e}{\hat{N} : M : \hat{N} + \frac{2}{3} K^I} \]

Using

\[ \hat{N} : M = 2 \mu \hat{N}^I, \quad \hat{N} : M : \hat{N} = 2 \mu, \quad \hat{N} : M : E = 2 \mu \hat{N} \cdot E \]

yields

\[ \gamma = \frac{\hat{N} \cdot \kappa^e}{K} \] (66)

Using this result into the generalized Hooke’s law yields
The constitutive tensor given by Eq. 67 is equivalent to the one in Eq. 62. It can be seen from Eq. 62 that the coupling between strains and curvatures becomes evident in the off-diagonal terms in the generalized constitutive tensor.

### 5.2 Radial return algorithm - Rate independent model - Linear isotropic hardening.

From Eq. 64 and after using a Backwards Euler difference scheme it follows that

\[ \hat{Y}_{n+1} = \hat{Y}_n + \Delta \dot{\gamma} \hat{\mathbf{N}}_{n+1} \]

\[ \alpha_{n+1} = \alpha_n + \frac{2}{3} \Delta \gamma \]  

(68)

Now consider the following trial state obtained after freezing plastic flow,

\[ \Sigma_{n+1}^\tau = \Sigma_n + 2 \mu \Delta E_{n+1} \]

\[ \Sigma_{n+1} = \Sigma_{n+1}^\tau - 2 \mu \Delta \gamma \hat{\mathbf{N}}_{n+1} \]  

(69)

from which it is concluded that

\[ \| \Sigma_{n+1} \| + 2 \mu \Delta \gamma = \| \Sigma_{n+1}^\tau \| \]  

(70a)

and

\[ \hat{\mathbf{N}}_{n+1} = \frac{\Sigma_{n+1}^\tau}{\| \Sigma_{n+1}^\tau \|} \]  

(70b)

Assuming linear isotropic hardening it follows from Eq. 70b

\[ \| \Sigma_{n+1}^\tau \| = \sqrt{\frac{2}{3} K (\alpha_n) - \frac{2}{3} K' \Delta \gamma - 2 \mu \Delta \gamma} = 0 \]

which is solved for the consistency parameter

\[ \Delta \gamma = \frac{F_{n+1}^\tau}{2 \mu \bar{K}} \]  

(71)
Linearization

Consider

\[ \mathbf{X}_{n+1} = M : \mathbf{X}_{n+1} - 2\mu \Delta \gamma \hat{N}_{n+1} \]

then it follows that

\[ \frac{d\Sigma_{n+1}}{dE_{n+1}} = M - 2\mu \left[ \hat{N}_{n+1} \otimes \frac{d\Delta \gamma}{dE_{n+1}} + \Delta \gamma \frac{d\hat{N}_{n+1}}{dE_{n+1}} \right] \]

using

\[ \frac{d\Delta \gamma}{dE_{n+1}} = \frac{1}{K} \hat{N}_{n+1} \]

and

\[ \frac{d\hat{N}_{n+1}}{dE_{n+1}} = \frac{2\mu}{\|\Sigma_{n+1}\|} H_{n+1} \]

where

\[ H_{n+1} = \begin{bmatrix} \hat{C} - \frac{1}{3} \mathbf{I} \otimes \mathbf{I} - \hat{n}_{n+1} \otimes \hat{n}_{n+1} & 0 \\ 0 & \hat{C} - \hat{\mathbf{v}}_{n+1} \otimes \hat{\mathbf{v}}_{n+1} \end{bmatrix} \]

thus

\[ \frac{d\Sigma_{n+1}}{dE_{n+1}} = M - 2\mu \left[ \frac{\hat{N}_{n+1} \otimes \hat{N}_{n+1}}{K} + \frac{\Delta \gamma}{\|\Sigma_{n+1}\|} \frac{2\mu}{H_{n+1}} \right] \]  \hspace{1cm} (72)

Expanding Eq.4-88 yields

\[ M_{n+1}^{EP} = \begin{bmatrix} \kappa \mathbf{I} \otimes \mathbf{I} + 2\mu \delta_{n+1} \left( \hat{C} - \frac{1}{3} \mathbf{I} \otimes \mathbf{I} \right) - 2\mu \tilde{\mathbf{v}}_{n+1} \hat{\mathbf{v}}_{n+1} \otimes \hat{\mathbf{v}}_{n+1} & \frac{\Delta \gamma}{K} \frac{2\mu}{\|\Sigma_{n+1}\|} \hat{n}_{n+1} \otimes \hat{n}_{n+1} \\ \frac{\Delta \gamma}{K} \frac{2\mu}{\|\Sigma_{n+1}\|} \hat{n}_{n+1} \otimes \hat{n}_{n+1} & 2\mu \delta_{n+1} \hat{C} - 2\mu \tilde{\mathbf{v}}_{n+1} \mathbf{v}_{n+1} \otimes \hat{\mathbf{v}}_{n+1} \end{bmatrix} \]  \hspace{1cm} (73)
when $\Delta t \to 0$ Eq.73 tends to Eq.62. The coupling between the curvatures and strain apparent in the continuous version of the tangent stiffness matrix is again evident in the algorithmic version as can be seen from Eq.73. The above algorithm has been implemented in the material subroutine labeled UMAT in the general subroutine described in Box 1.

### 6.0 Numerical validation- simulation of the microbending experiment.

The computational framework proposed in the present article is validated against the results corresponding to the microbending experiments from Stolken and Evans(1998) and Shrotiya et al(2003). In both experiments the loads are applied monotonically and the material is assumed to exhibit rate independent behavior. The material properties, including the values of the length scale parameter $\ell$ correspond to the ones reported in the original papers. The thickness of the microbeams in Stolken and Evans(1998) correspond to $12.5\,\mu m$, $25.0\,\mu m$ and $50.0\,\mu m$ and those in Shrotiya et al(2003) correspond to $25.0\,\mu m$ and $50.0\,\mu m$ and $100.0\,\mu m$. For the $100.0\,\mu m$ specimens the gradient effects vanish and are not included here. Figure 8 shows the used finite element mesh for the numerical simulations. The material parameters corresponding to both tests are reported in Table 1. In the simulations a point load was statically applied at the tip of the different microbeams in order to achieve a specified moment. Figures 9 and 10 show the experimental and numerical simulation results for both tests. In Fig 9 which corresponds to the Stolken and Evans(1998) test the normalized moment $\frac{4M}{\Sigma_{0}bt^{2}}$ is meaningful within the Fleck and Hutchinson(1997) strain gradient plasticity theory. In both simulations the used length scale parameters are $\ell = 5.0\,\mu m$ and $\ell = 5.6\,\mu m$. The simulation results from the Stolken and Evans(1998) test are in very good agreement with the experiments. However this is
not the case for the simulations results shown in Figure 10 of the test by Shrotiya et al(2003). Trying to explain this difference we compared the experimental results corresponding to the tests by both set of authors in Figure 11. It is clear that the results from the experiments by Shrotiya et al(2003) are not as consistent as the results from Stolken and Evans(1998). Apparently there is some error not reported in those experiments. Figure 12 shows computational results for the Stolken and Evans(1998) test using a value of $\ell = 0.0\mu m$ which correspond to classical theory. In the same Figure the corresponding experimental results are also shown for comparison. From the comparison with the experimental results from both set of authors we conclude that in general there is very good agreement between the simulation and the experiments.

7.0 Conclusions.

The numerical treatment of a commonly used strain gradient plasticity, namely the Cosserat couple stress theory, has been discussed. Two main issues are identified from a numerical point of view. First is the need for higher order continuity requirements on the interpolation shape functions which is demanded by the presence of the gradients of strain. Second is the need for an integration algorithm when the theory is casted in its rate independent flow theory form. To study the first problem the theory has been placed into a variational approach. As a result different alternatives for the finite element formulation of the problem become available. Although some of these different alternatives have been previously used by other authors we have placed them here within a unified mathematical framework. In particular we have implemented the reduced Cosserat couple stress theory in the form of a reduced integration/penalty function scheme in the form of a user element in the commercial finite element code ABAQUS. The Cosserat theory has been further extended to consider nonlinear
material behavior. The equations for the case of a rate independent material have been presented in its flow theory form. It is clear that once plastic behavior takes place there is coupling between the normal stresses and the couple stresses quantities. The rate independent constitutive model has been complemented with an integration algorithm thus allowing its implementation in the user element subroutine UEL. In order to validate the implementation we have used the microbending test in thin nickel foils from Stolken and Evans (1998) and Shrotriya et al (2003). Comparisons between the simulations and the experimental results shows general good agreement.
Figure 1. Schematic of general solid
Figure 2. Relative deformation in the Couple stress theory by Mindlin (1964)
Figure 3. Kinematics of the reduced theory Couple stress solid.

Figure 4. Typical finite element for the case of the general couple stress theory.
Figure 5. Typical finite element for the case of reduced couple stress theory using translational degrees of freedom only.

Figure 6. Typical finite element for the case of reduced couple stress theory using Lagrange multipliers.
Figure 7. Typical finite element for the case of reduced couple stress theory using Penalty based approach.
Figure 8. Finite element mesh of microbeam used in the numerical simulations.
Figure 9  Stolken and Evans(1998) Microbending Experimental Results on Nickel Foils Compared to Present Model.

\[ l = 5.0\mu m, E = 220.0\text{GPa}, \Sigma_0 = 103\text{MPa} \]
Figure 10. Shrotriya et al (2003) Microbending Experimental Results on LIGA Nickel Foils Compared to Present Model.

\[ l = 5.6 \mu m, E = 220.0 GPa, \Sigma_0 = 400 MPa \]
Comparisson of Microbending Test Results

Figure 11. Experimental results from the microbending tests on LIGA-Nickel foils and pure Nickel foils from Shoritaya et al(2003) and Stolken and Evans(1998)
Figure 12. Analysis of Stolken and Evans (1998) Specimens Using a Classical Plasticity Theory Model. The classical theory results are compared against experimental results.

\[ l = 0.0 \mu m, E = 220.0 GPa, \Sigma_o = 103 MPa \]
Table 1. Material parameters-microbending test simulation.

<table>
<thead>
<tr>
<th>Stolken and Evans (1998)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Beam thickness(μm)</td>
<td>Σ₀ (Mpa)</td>
</tr>
<tr>
<td>12.50</td>
<td>56.00</td>
</tr>
<tr>
<td>25.00</td>
<td>75.00</td>
</tr>
<tr>
<td>50.00</td>
<td>103.00</td>
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</table>

<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>25.00</td>
<td>400.00</td>
</tr>
<tr>
<td>50.00</td>
<td>400.00</td>
</tr>
</tbody>
</table>
Box 1 Newton-Raphson iteration in ABAQUS (UEL).

Let \( T \leftarrow 0 \) (time)

1 Assume \( 'u, 'E, '\Sigma, '\kappa \) known.

2 Assemble \( t+\Delta F_{ext} = \int_S N^T(t+\Delta \eta)dS \)

Initialize \( (t+\Delta u)^{(0)} \leftarrow 'u \) and \( (t+\Delta E)^{(0)} \leftarrow 'E \)

Let \( i \leftarrow 0 \), \( Flag \leftarrow 0 \)

**Do_While** \( Flag = 0 \)

\( i \leftarrow i + 1 \)

(ABAQUS calls user subroutine UEL.f)

Assemble \( B_e, B_a \)

Call UMAT.f to compute \( t+\Delta \Sigma(i-1), t+\Delta \kappa(i-1), t+\Delta C(i-1), t+\Delta D(i-1) \)

Assemble \( t+\Delta K(i-1) \)

Assemble residual (RHS)

\( t+\Delta F((i-1)) \leftarrow t+\Delta F_{ext} - \int_V B_e^T t+\Delta \Sigma(i-1) dV - \int_V B_a^T t+\Delta \eta(i-1) dV \)

(Exits user subroutine UEL.f and returns to ABAQUS)

Solve \( t+\Delta K(i-1) \Delta u(i) = t+\Delta F((i-1)) \)

Update

\( t+\Delta u(i) \leftarrow t+\Delta u(i-1) + \Delta u(i) \)

\( t+\Delta E(i) \leftarrow B(t+\Delta E(i)) \)

**If** \( \|error\| < tol \) **Then** \( Flag \leftarrow 1 \)

**End_Do_While**

Let \( T \leftarrow T + \Delta T \)

**If** \( T < T_{\text{max}} \) **Then** Go to step 2

Analysis Complete
REFERENCES


Gutierrez De La Merced, MA , & Borst, R de (2002). Deterministic and probabilistic


